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ABSTRACT

Multiple ratio goals often occur in equity considerations and imply non-linear non-convex programming problems. Charnes and Cooper's solution of such a class by a sequence of linear programs is herein implemented by an effective computational method applicable to a larger class of non-linear non-convex problems. An analytical prescription of the recursion to preserve feasibility of the last basic solution is derived. A new usage of the fast re-inversion routine of current LP codes to obtain the modified basis inverse is developed together with the further subroutines of the Opposite Sign Algorithm of Charnes and Kortanek (1963) or a negative image scheme whenever the old basic solution is not basic feasible for the modified problem. An illustration in terms of funding of State colleges in Texas is provided.

KEY WORDS

Non-convex programming

Ratio goals programming

Multi-objective programming

Equitable funds allocation

1. INTRODUCTION

Charnes and Cooper in [1] consider the non-linear non-convex ratio goals problem:

(1) $$\min \lambda$$

$$\text{with} \quad \lambda \geq \left| \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i} - \rho_i \right| \quad , i = 1,\ldots,r$$

$$\text{and} \quad Ax = b \quad , x \geq 0$$

where c_i, d_i, b are column vectors, T denotes transpose, α_i, β_i, ρ_i are scalars with ρ_i the ratio goals, A an m x n matrix, x an n x 1 column vector and $d_i^T x + \beta_i > 0$ for all i and admissible x.

They set up the LP problem equivalent to:

(2) $$\max \nu$$

$$\text{with} \quad \nu \leq \bar{\lambda}\,(d_i^T x + \beta_i) - |c_i^T x + \alpha_i - \rho_i(d_i^T x + \beta_i)|$$

$$\text{and} \quad Ax = b \quad , x \geq 0$$

where $\bar{\lambda}$ is a posited upper limit for the ratio goal deviations.

Then, associating feasible solutions $\bar{x}$ with (2) and λ,x with (1), and using asterisks to denote optimal values, the following possibilities are at hand.

(i) $\nu^* < 0$; $\bar{\lambda},\bar{x}^*$ is not feasible for (1).

(ii) $\nu^* = 0$; $\bar{\lambda} = \lambda^*$ and $\bar{x}^*$ is optional for (1).

(iii) $\nu^* > 0$; $\bar{\lambda},\bar{x}^*$ is feasible for (1) with

$\lambda^* \leq \lambda^*(\bar{x}^*) < \bar{\lambda}$, an improvement over $\bar{\lambda}$,

where

$$\lambda^*(\bar{x}^*) = \max_i \left\{ \left| \frac{c_i^T \bar{x}^* + \alpha_i}{d_i^T \bar{x}^* + \beta_i} - \rho_i \right| \right\}$$

and λ^* is optimal for (1).

In case (i), $\bar{\lambda}$ is too small to be attained. The predetermined value must be recast higher and the new linear program solved.

In case (ii), we are done.

In case (iii), we replace $\bar{\lambda}$ by λ^* $(\bar{x}^*)$ and proceed as before. Continuing in this manner we set up a simple sequence of linear programming problems which

converges to a solution of the nonconvex programming problem (1).

For effective solution, however, we must have an efficient automatic way to continue from LP problem to LP problem without starting from scratch and to choose the next value $\bar{\bar{\lambda}}$ to replace $\bar{\lambda}$. We show how to maintain feasibility from LP problem to LP problem by choice of $\bar{\bar{\lambda}}$. Employing the fast reinversion capability of current LP software (a first such usage) and either the Opposite Sign Algorithm of Charnes and Kortanek [3] or an automatic negative image method, we achieve our objective. It is illustrated with a problem of Charnes, Cox and Lane [2] for allocating State funds to colleges in Texas.

2. FEASIBILITY PRESERVATION

We need treat only case (iii). We determine the minimum $\bar{\bar{\lambda}}$ so that $\bar{x}^*$ is feasible. The only constraints to be considered are those involving ν. Writing these as linear inequalities in $\bar{\bar{\lambda}}$, $\bar{x}^*$, $\bar{\lambda}$ and setting $\nu \equiv \bar{\nu}^* + \delta$, we obtain:

$$(3)\qquad \delta + \bar{\nu}^* - [\bar{\lambda} d_i^T - (c_i^T - \rho_i d_i^T)]\bar{x}^* + (\bar{\lambda} - \bar{\bar{\lambda}}) d_i^T \bar{x}^* \leq \bar{\lambda}\beta_i - (\alpha_i - \rho_i \beta_i) - (\bar{\lambda} - \bar{\bar{\lambda}})\beta_i$$

$$\delta + \bar{\nu}^* - [\bar{\lambda} d_i^T + (c_i^T - \rho_i d_i^T)]\bar{x}^* + (\bar{\lambda} - \bar{\bar{\lambda}}) d_i^T \bar{x}^* \leq \bar{\lambda}\beta_i + (\alpha_i - \rho_i \beta_i) - (\bar{\lambda} - \bar{\bar{\lambda}})\beta_i$$

for $i = 1, \ldots, r$

Feasibility requires both

$$(4)\qquad \delta \geq -\bar{\nu}^* \quad \text{and} \quad \delta + (\bar{\lambda} - \bar{\bar{\lambda}}) d_i^T \bar{x}^* \leq -(\bar{\lambda} - \bar{\bar{\lambda}})\beta_i, \; i = 1, \ldots, r$$

or,

$$(4.1)\qquad (\bar{\lambda} - \bar{\bar{\lambda}})[d_i^T \bar{x}^* + \beta_i] \leq -\delta \leq \bar{\nu}^* \;, \; i = 1, \ldots, r$$

or,

$$(4.2)\qquad (\bar{\lambda} - \bar{\bar{\lambda}}) \leq \frac{\bar{\nu}^*}{\Delta(\bar{\lambda})} \qquad \text{where} \quad \Delta(\bar{\lambda}) = \max_i \, [d_i^T \bar{x}^* + \beta_i]$$

Thus,

$$(5)\qquad \bar{\bar{\lambda}} = \bar{\lambda} - \frac{\bar{\nu}^*}{\Delta(\bar{\lambda})}$$

provides the minimum $\bar{\bar{\lambda}}$, i.e., the maximum feasible change from $\bar{\lambda}$ preserving feasibility of $\bar{x}^*$.

3. BASIC FEASIBILITY VIOLATION

Although $\bar{x}^*$ is feasible for $\bar{\bar{\lambda}}$ and is a basic solution for $\bar{\lambda}$, it may fail to

be a basic feasible solution for $\bar{\bar{\lambda}}$, i.e., positive slack variables may occur in (the equation form of) the "ν" constraints. Generally, we expect few of these slacks to occur. Since the $\bar{x}^*$ coefficient vectors form a basis for the $\bar{\bar{\lambda}}$ constraint vectors and we have a feasible solution, we can employ the (1963) Opposite Sign Algorithm of Charnes and Kortanek [3] coded in ALGOL by Kortanek and Raike [4].

The change from $\bar{\lambda}$ to $\bar{\bar{\lambda}}$ involves a change (generally small) in the coefficient vectors (in the "ν" conditions part). We, therefore, treat our basic inverse for $\bar{\lambda}$ as an approximate inverse for $\bar{\bar{\lambda}}$ and call the fast re-inversion routine to give us an accurate basis inverse for $\bar{\bar{\lambda}}$. Such a procedure, heretofore not utilized by anyone, can be employed in many other non-linear problems treated parametrically.

An alternate to the Opposite Sign Algorithm is simply to immediately introduce the negatives of the coefficient vectors of $\bar{x}^*$ for which $\bar{x}_i^* < 0$ as artificials with high penalties and to continue onwards. We are currently experimenting with both stratagems.

4. MODELS FOR EQUITABLE ALLOCATION OF STATE FUNDS

The State of Texas has n institutions of higher education, and a total of m subjects (e.g., engineering, economics, business, etc.) are taught in these institutions (not necessarily all in each). The existing system of allocation is to multiply a set rate for each subject area (R_j, j=1, ..., m) by the number of Semester Credit Hours (SCH) produced in institution i in subject j, N_{ij}, and sum. Thus, institution i receives $\sum_{j=1}^{m} R_j N_{ij} = T_i$ and the total funds awarded to these institutions is

$$\sum_{i=1}^{r} \sum_{j=1}^{m} R_j N_{ij} = \sum_{i=1}^{r} T_i = T.$$

The introduction of two new institutions which will only have upper division courses into the system also increased the need for a new system of rates. It was desired that:

1. The rates should more accurately reflect actual costs.
2. No institution should receive less under the new system than under the old.
3. The same rate should apply at all institutions for each subject area.
4. The extra amount ("institutional excess") that any institution receives under the new system should be minimized.

Charnes et al. set up the following LP model to minimize the maximum institutional excess subject to constraints which approximately represented in linear inequality fashion the desired ratio goals:

(6.1) $\quad \text{Min } \lambda$

Subject to

(6.2) $$y_i \le \lambda \qquad , i = 1,\ldots,r$$

(6.3) $$\textstyle\sum_{j=1}^{m} R_j^{(L)} N_{ij}^{(L)} + \sum_{j=1}^{m} R_j^{(U)} N_{ij}^{(U)} - y_i = T_i \qquad , i = 1,\ldots,r$$

(6.4) $$(1-\bar{\alpha}) \frac{C_j^{(L)}}{C_1^{(L)}} \le \frac{R_j^{(L)}}{R_1^{(L)}} \le (1+\bar{\alpha}) \frac{C_j^{(L)}}{C_1^{(L)}} \quad , j = 2,\ldots,m$$

(6.5) $$(1-\bar{\alpha}) \frac{C_j^{(U)}}{C_1^{(U)}} \le \frac{R_j^{(U)}}{R_1^{(U)}} \le (1+\bar{\alpha}) \frac{C_j^{(U)}}{C_1^{(U)}} \quad , j = 2,\ldots,m$$

(6.6) $$(1-\bar{\beta}) \frac{C_j^{(U)}}{C_j^{(L)}} \le \frac{R_j^{(U)}}{R_j^{(L)}} \le (1+\bar{\beta}) \frac{C_j^{(U)}}{C_j^{(L)}} \quad , j = 1,\ldots,m$$

$$y_i, \lambda, R_j^{(U)}, R_j^{(L)} \geq 0 \qquad \begin{array}{l} j = 1,\ldots,m \\ i = 1,\ldots,r \end{array}$$

where $R_j^{(L)}, R_j^{(U)}$ = the rate for lower (upper) division SCH's in subject j.

$N_{ij}^{(L)}, N_{ij}^{(U)}$ = the number of SCH's produced by institution i in subject area j, at lower (upper) division levels.

$C_j^{(L)}, C_j^{(U)}$ = the estimated cost of subject area j at lower (upper) division levels.

y_i = the institutional excess for institution i.

$\bar{\alpha}, \bar{\beta}$ = the allowable deviation of ratios from lower (upper) division goals.

In ratio goals form, our model is:

Min λ

(7) $$\lambda \geq \left| \frac{\sum_{j=1}^{m} R_j^{(L)} N_{ij}^{(L)} + \sum_{j=1}^{m} R_j^{(U)} N_{ij}^{(U)}}{\sum_{j=1}^{m} R_j^{(L)} N_{1j}^{(L)} + \sum_{j=1}^{m} R_j^{(U)} N_{1j}^{(U)}} - \frac{T_i}{T_1} \right| \quad , \qquad i = 2,\ldots,r$$

$$\lambda \geq \left| \frac{R_j^{(L)}}{R_1^{(L)}} - \frac{C_j^{(L)}}{C_1^{(U)}} \right| , \left| \frac{R_j^{(U)}}{R_1^{(U)}} - \frac{C_j^{(U)}}{C_1^{(U)}} \right| \qquad j = 2,\ldots,m$$

$$\lambda \geq \left| \frac{R_j^{(U)}}{R_j^{(L)}} - \frac{C_j^{(U)}}{C_j^{(L)}} \right| \qquad j = 1,\ldots,m$$

$$\sum_{j=1}^{m} R_j^{(L)} N_{ij}^{(L)} + \sum_{j=1}^{m} R_j^{(U)} N_{ij}^{(U)} \geq T_i \qquad i = 1,2,\ldots,r$$

with $R_j^{(L)}, R_j^{(U)} \geq 0.$

5. NUMERICAL EXAMPLE

For the purpose of illustration, four institutions and three subject areas are selected. With the exception of SCH's and resulting current allocation, all data are from [2].

COSTS

Subject Area (j)	Existing Rate (R_j)	Lower Division ($C_j^{(L)}$)	Upper Division ($C_j^{(U)}$)
1	15.12	9.49	16.91
2	16.49	8.44	22.87
3	27.57	18.78	33.57

SCH'S

Subj. \ Inst.	1 L	1 U	2 L	2 U	3 L	3 U	4 L	4 U
1	1345	860	980	860	2150	170	750	-
2	210	190	800	340	650	-	500	-
3	1145	950	-	-	2000	380	1250	-
TOTAL	3000	2000	1780	1200	4800	550	2500	-

CURRENT ALLOCATION OF STATE FUNDS

Subj. \ Inst.	1	2	3	4
1	33,339.6	27,820.8	35,078.4	11,340
2	6,596.0	18,798.6	10,718.5	8,245
3	66,030.15	-	65,616.6	34,462.5
TOTAL	105,965.75	46,619.4	111,413.5	54,047.5

RESULTS:

When the problem is formulated as in (6) and solved, the following results are obtained.

Subject Area (j)	Existing Rate (R_j)	Goals Ratio of Costs ($C_j^{(U)}/C_j^{(L)}$)	New Rates Lower Division ($R_j^{(L)}$)	New Rates Upper Division ($R_j^{(U)}$)	Goal Achievements Ratio of Rates ($R_j^{(U)}/R_j^{(L)}$)
1	15.12	1.78	12.73	18.07	1.42
2	16.49	2.71	13.49	29.28	2.17
3	27.57	1.73	30.21	43.19	1.43

Goals	Achievement
$\frac{C_2^{(L)}}{C_1^{(L)}} = 0.89$	$\frac{R_2^{(L)}}{R_1^{(L)}} = 1.05$
$\frac{C_3^{(L)}}{C_1^{(L)}} = 1.98$	$\frac{R_3^{(L)}}{R_1^{(L)}} = 2.37$
$\frac{C_2^{(U)}}{C_1^{(U)}} = 1.35$	$\frac{R_2^{(U)}}{R_1^{(U)}} = 1.62$
$\frac{C_3^{(U)}}{C_1^{(U)}} = 1.98$	$\frac{R_3^{(U)}}{R_1^{(U)}} = 2.39$

INSTITUTIONAL EXCESS:

Institution	Excess
1	$19,770.20
2	2,143.25
3	4,615.16
4	0
	$26,528.61

When the formulation of (7) is used and a total excess of $25,954 is allowed (about $1,000 less than the excess resulted from the LP solution), the following results are obtained.

Subject Area (j)	Existing Rate (R_j)	Goals Ratio of Costs ($C_j^{(U)}/C_j^{(L)}$)	New Rates Lower Division ($R_j^{(L)}$)	New Rates Upper Division ($R_j^{(U)}$)	Goal Achievements Ratio of Rates ($R_j^{(U)}/R_j^{(L)}$)
1	15.12	1.78	13.94	21.97	1.58
2	16.49	2.71	11.09	30.04	2.71
3	25.57	1.73	30.44	35.23	1.16

Goals	Achievement
$\frac{C_2^{(L)}}{C_1^{(L)}} = 0.89$	$\frac{R_2^{(L)}}{R_1^{(L)}} = 0.80$
$\frac{C_3^{(L)}}{C_1^{(L)}} = 1.98$	$\frac{R_3^{(L)}}{R_1^{(L)}} = 2.18$
$\frac{C_2^{(U)}}{C_1^{(U)}} = 1.35$	$\frac{R_2^{(U)}}{R_1^{(U)}} = 1.37$
$\frac{C_3^{(U)}}{C_1^{(U)}} = 1.98$	$\frac{R_3^{(U)}}{R_1^{(U)}} = 1.60$

INSTITUTIONAL EXCESS:

Institution	Excess
1	$17,167.86
2	5,021.93
3	3,764.06
4	0
	$25,953.85

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Multiple ratio goals often occur in equity considerations and imply non-linear non-convex programming problems. Charnes and Cooper's solution of such a class by a sequence of linear programs is herein implemented by an effective computational method applicable to a larger class of non-linear non-convex problems. An analytical prescription of the recursion to preserve feasibility of the last basic solution is derived. A new usage of the fast re-inversion routine of current LP codes to obtain the modified basis inverse is developed together with the further subroutines of the Opposite Sign

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